## 4.6 - Variation of Parameters

Consider the homogeneous second-order differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. The general solution to this DE is $y = c_1y_1 + c_2y_2$. When the input function of an associated nonhomogeneous DE is not restricted to the type we saw in 4.4, we consider particular solutions with coefficients that are variable functions. That is, we will find $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.

$\ln x$, $\tan x$, $\frac{e^x}{x}$, $\sec x$

Starting with $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

Standard form: $y'' + P(x)y' + Q(x)y = f(x)$

complementary function from 4.3: $y_c = c_1y_1 + c_2y_2$

When $g(x)$ is not restricted like in 4.4, we consider variable coefficient functions:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$y_p'' = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$

$y'' + P(x)y' + Q(x)y = f(x)$ becomes

$$u_1''y_1 + 2u_1'y_1' + \cancel{u_1y_1''} + u_2''y_2 + 2u_2'y_2' + \cancel{u_2y_2''}$$
$$+ Pu_1'y_1 + \cancel{Pu_1y_1'} + Pu_2'y_2 + \cancel{Pu_2y_2'}$$
$$+ \cancel{Qu_1y_1} + \cancel{Qu_2y_2} = f(x)$$

$$u_1 y_1'' + P u_1 y_1' + Q u_1 y_1 = u_1 (y_1'' + P y_1' + Q y_1) = 0$$

Likewise for $u_2$

$$\underbrace{u_1'' y_1 + 2u_1' y_1'}_{u_1'' y_1 + u_1' y_1' + u_1' y_1' \; = \; \frac{d}{dx}(u_1' y_1) + u_1' y_1'} + \underbrace{u_2'' y_2 + 2u' y_2'}_{\text{contains } \frac{d}{dx}(u_2' y_2)} + P(u_1' y_1 + u_2' y_2) = f(x)$$

becomes $\frac{d}{dx}(u_1' y_1 + u_2' y_2) + u_1' y_1' + u_2' y_2'$
$+ P(u_1' y_1 + u_2' y_2) = f(x)$

If we require that $u_1' y_1 + u_2' y_2 = 0$,
then $u_1' y_1' + u_2' y_2' = f(x)$

this has the form of
$$ax + by = e$$
$$cx + dy = f$$

where $x = u_1'$, $y = u_2'$ are the unknown functions.

This can be solved using Cramer's Rule:

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad D_x = \begin{vmatrix} e & b \\ f & c \end{vmatrix}, \quad D_y = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

Here, we have $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$,

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Requires the DE to be in Std form

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_1 = -f(x)\,y_2 \qquad W_2 = y_1 f(x)$$

$$u_1' = \frac{W_1}{W} \implies u_1 = \int \frac{W_1}{W}\,dx$$

$$u_2' = \frac{W_2}{W} \implies u_2 = \int \frac{W_2}{W}\,dx$$

$y_p = u_1 y_1 + u_2 y_2$ and $y = y_c + y_p$

**Example:** Solve each differential equation by variation of parameters.

$y'' + y = \sec\theta\ \tan\theta$ — Std form ✓

Homog: $m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c = c_1 \cos\theta + c_2 \sin\theta$

so $y_1 = \cos\theta$, $y_2 = \sin\theta$

$$W = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin\theta \\ \sec\theta\tan\theta & \cos\theta \end{vmatrix} = -\tan^2\theta$$

$$W_2 = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \sec\theta\tan\theta \end{vmatrix} = \tan\theta$$

$$u_1 = \int \frac{W_1}{W}\,d\theta = \int -\tan^2\theta\,d\theta = \int (1-\sec^2\theta)\,d\theta$$

$\sec^2\theta - 1$

$$= \theta - \tan\theta$$

$$u_2 = \int \frac{W_2}{W}\,d\theta = \int \tan\theta\,d\theta = \ln|\sec\theta|$$

$$-\int \frac{-\sin\theta}{\cos\theta}\,d\theta = -\ln|\cos\theta|$$

$$y_p = u_1 y_1 + u_2 y_2 = (\theta - \tan\theta)\cos\theta + \ln|\sec\theta|\sin\theta$$

$$= \theta\cos\theta - \sin\theta + \ln|\sec\theta|\sin\theta$$

already in $y_c$

$$\boxed{y = c_1\cos\theta + c_2\sin\theta + \theta\cos\theta + \sin\theta\ln|\sec\theta|}$$

**Example:** Solve the initial-value problem by variation of parameters.

$2y'' + y' - y = x + 1 \qquad y(0) = 1,\ y'(0) = 0$

$$y'' + \tfrac{1}{2}y' - \tfrac{1}{2}y = \tfrac{1}{2}x + \tfrac{1}{2}$$

$$2m^2 + m - 1 = 0 \Rightarrow (2m-1)(m+1) = 0$$

$$m = 1/2,\ -1 \Rightarrow y_c = c_1 e^{1/2 x} + c_2 e^{-x}$$

$$y_1 = e^{1/2 x} \qquad y_2 = e^{-x}$$

$$W = \begin{vmatrix} e^{1/2x} & e^{-x} \\ \frac{1}{2}e^{1/2x} & -e^{-x} \end{vmatrix} = -e^{-1/2x} - \frac{1}{2}e^{-1/2x} = -\frac{3}{2e^{1/2x}}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{2}x+\frac{1}{2} & -e^{-x} \end{vmatrix} = -\frac{x+1}{2e^{x}} \Rightarrow \frac{W_1}{W} = \frac{(x+1)e^{-1/2x}}{3}$$

std form first

$$W_2 = \begin{vmatrix} e^{1/2x} & 0 \\ \frac{1}{2}e^{1/2x} & \frac{1}{2}x+\frac{1}{2} \end{vmatrix} = \frac{(x+1)e^{1/2x}}{2} \Rightarrow \frac{W_1}{W} = -\frac{(x+1)e^{x}}{3}$$

$$u_1 = \int \frac{W_1}{W}dx = \frac{1}{3}\int (x+1)e^{-1/2x}dx$$

$u = x+1$

$dv = e^{-1/2x}dx$

$$u_2 = \int \frac{W_2}{W}dx = -\frac{1}{3}\int (x+1)e^{x}dx$$

After integrating by parts twice we get

$$y = c_1e^{1/2x} + c_2e^{-x} - x - 2$$

$y(0)=1,\ y'(0)=0$

$y(0)=1:\ 1 = c_1 + c_2 - 2$

$y'(0) = \frac{1}{2}c_1e^{1/2x} - c_2e^{-x} - 1 \Rightarrow y'(0)=0:\ 0 = \frac{1}{2}c_1 - c_2 - 1$

$c_2 = \frac{1}{3},\ c_1 = \frac{8}{3}$ so

$$y = \frac{8}{3}e^{x/2} + \frac{1}{3}e^{-x} - x - 2$$

**Example:** Solve the differential equation by variation of parameters.

$y'' - 4y = \frac{e^{2x}}{x}$ Std form ✓

$m^2 - 4 = 0 \Rightarrow m = \pm 2$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$

$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = -\frac{1}{x}$

$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{e^{4x}}{x}$

assuming $x > 0$

$u_1 = \int \frac{W_1}{W} dx = \int \frac{1}{4x} dx = \frac{1}{4} \ln x$

$u_2 = \int \frac{W_2}{W} dx \int - \frac{e^{4x}}{4x} dx$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4} e^{2x} \ln x - e^{-2x} \int_{x_0}^{x} \frac{e^{4t}}{4t} dt}$$

**Example:** Solve the given third-order differential equation by variation of parameters.
$y''' + 4y' = \sec 2x$

$$\rightarrow \quad y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$u_1 = \int \frac{W_1}{W}\,dx \quad \ldots$$